Exam Lie Groups in Physics

Date	November 5, 2018
Room	BB 5161.0165
Time	14:00 - 17:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	8	2a)	8	3a)	8	4a)	8
1b)	8	2b)	8	3b)	8	4b)	8
1c)	6	2c)	6			4c)	6
1d)	8						

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Problem 1

Consider the group $SO(1,1)$ of all 2×2 real matrices O that have determinant equal to 1 and satisfy

$$O^T = gO^{-1}g^{-1} \quad \text{with} \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Write down the general form of elements O in $SO(1,1)$ and show that such matrices form a non-compact Abelian group.

(b) Determine the connected components of $SO(1,1)$ and show that $SO(1,1)/\mathsf{Z}_2$ is isomorphic to the group $(\mathsf{R};+)$ of real numbers with addition as the composition law.

(c) Show whether the defining representation of $SO(1,1)$ is irreducible or not.

(d) Write down the corresponding representation of the Lie algebra of $SO(1,1)$ and show whether it is an irrep of the Lie algebra.

Problem 2

Consider the group $O(3)$ of real orthogonal 3×3 matrices.

(a) Show the isomorphism $O(3) \cong \mathsf{Z}_2 \otimes SO(3)$.

(b) Show that the symmetric and antisymmetric tensors $x_i y_j \pm x_j y_i$ do not mix under $O(3)$ transformations.

(c) Argue that the defining representation of $O(3)$ is irreducible and becomes reducible when restricting to an $O(2)$ subgroup.

Problem 3
Consider the Cartan matrix $A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$, where the α_i denote the simple roots, for the complex Lie algebra $\tilde{so}(5)$:

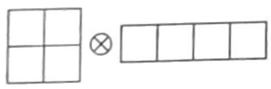
$$\boldsymbol{A} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

(a) Use this matrix to obtain the root diagram of $\tilde{so}(5)$ by using Weyl reflections and deduce the dimension of the Lie algebra $\tilde{so}(5)$.

(b) Count the dimension of the Lie group $SO(5)$ by using properties of its defining representation and compare the answer to the one obtained in part (a).

Problem 4
Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(3)$ and $su(4)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

(c) Consider for $su(3)$ the complex conjugate of the above direct product in terms of Young tableaux, decompose it into irreps and compare the answer to the one obtained in part (b).